A few results we need.

Archimedian Principle (Archie)

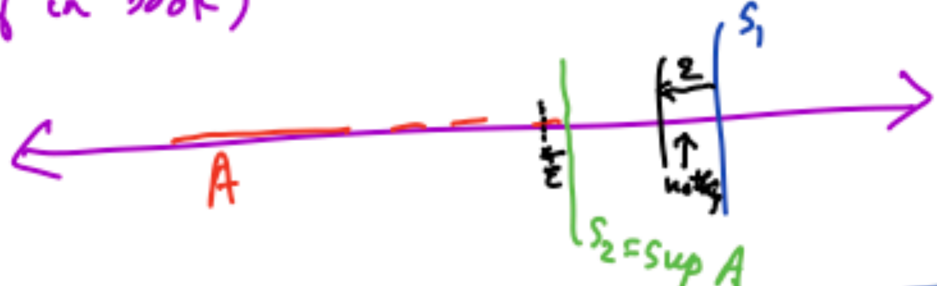
Two versions: (1) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $n > x$.

(2) $\forall y > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < y$.

Sup Lemma Let s be an upperbound of the nonempty set $A \subseteq \mathbb{R}$. Then

$s = \sup A \iff \forall \varepsilon > 0, \exists a \in A$ s.t.
$s - \varepsilon < a \leq s$

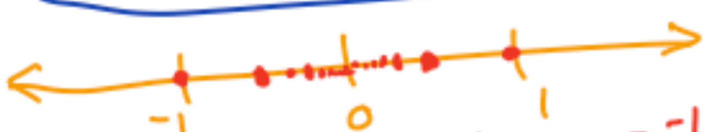
(Proof in book)



Example 1.3.8 (b) Find sup & inf, if they exist:

$\left\{ \frac{(-1)^m}{n} : m, n \in \mathbb{N} \right\} = B$

Scratch Work



sup = 1 inf = -1.

<u>Proof:</u> Observe that $\forall\, m, n \in \mathbb{N}$,

$$-1 \le \frac{-1}{n} \le \frac{(-1)^m}{n} \le \frac{1}{n} \le 1.$$

Thus, -1 is a lower bound for B, and 1 is an upper bound for B.

If $y < 1$, then since $1 \in B$, y is not an upper bound of B. Thus every upper bound z of B satisfies $z \ge 1$. $\therefore\ 1 = \sup B$.

If $y > -1$, then since $-1 \in B$, y is not a lower bound for B. Thus every lower bound p of B satisfies $p \le -1$. $\therefore\ -1 = \inf B$. $\square$

1.3.8(c) Find inf & sup if they exist.

$$C = \left\{ \frac{n}{3n+1} : n \in \mathbb{N} \right\}.$$

<u>Scratch.</u>

Guessing

$\inf = \frac{1}{4}$

$\sup = \frac{1}{3}$.

$$\frac{1}{4} = \frac{n}{3n+n} \le \frac{n}{3n+1} \le \frac{n}{3n} = \frac{1}{3}.$$

$\frac{1}{3} - \frac{n}{3n+1}$ should be a very small $+$ #.

$$\frac{3n+1}{3(3n+1)} - \frac{3n}{3(3n+1)} = \frac{1}{3(3n+1)} > 0$$ Yes!

$\leftarrow$ by archimedean we can make $< \varepsilon$

Actual Proof: $\forall n \in \mathbb{N}$, $\frac{1}{4} = \frac{n}{3n+n} \leq \frac{n}{3n+1} \leq \frac{n}{3n} = \frac{1}{3}$,

so that $\frac{1}{4}$ is a lower bound for C and $\frac{1}{3}$ is an upper bound for C.

Since $\frac{1}{4} = \frac{(1)}{3(1)+1} \in C$, any larger number would not be a lower bound for C. So if y is a lower bd for C, $y \leq \frac{1}{4}$. Thus, $\frac{1}{4} = \inf C$.

Wait

$$\frac{1}{3} - \varepsilon < \frac{n}{3n+1} \iff \underbrace{\frac{1}{3} - \frac{n}{3n+1}}_{\frac{1}{3(3n+1)}} < \varepsilon$$

$$\frac{1}{3(3n+1)} < \varepsilon$$

By Archie, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{3(3n+1)} < \frac{1}{n} < \varepsilon$

Next, $\forall \varepsilon > 0$, By Archie $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$, and so $\frac{1}{3(3n+1)} < \frac{1}{n} < \varepsilon$ also.

Now, consider $x = \frac{n}{3n+1} \in C$.

Then $\frac{1}{3} - \varepsilon < \frac{1}{3} - \frac{1}{3(3n+1)} = \frac{3n+1-1}{3(3n+1)}$

$$= \frac{3n}{3(3n+1)} = \frac{n}{3n+1} = x.$$

That is, $\frac{1}{3} - \varepsilon < x$.

By the sup lemma, $\frac{1}{3} = \sup C$. $\blacksquare$

Example: (Ex 1.3.2 in book)

(a) Find a set B such that $\inf(B) \geq \sup(B)$ (or show no such set exists.)

Ans: Let $B = \{1\}$.

Then $\inf B = 1$ (it's a lower bound, since $1 \leq 1$, and if $y > 1$, then y is not a lower bound. So if z is a lower bound of B, $z \leq 1$.).

Similarly $\sup B = 1$.

So $\inf B \geq \sup B$ in this case!

(b) Find a finite set C that contains its inf but not its sup. (Or show there is no such set.)

([3, 4) would be an example, but it's not finite.)

Thoughts:

No such C, because

$\sup C$ = greatest element of $C = x$.

x is definitely an upper bound,

Pf. There is no such set. If C is finite and nonempty, let x = greatest element of C. Then $\forall c \in C$,

$c \le x$, so x is an upper bound. If $y < x$, then y can't be an upper bound for all of C. Thus every upper bound z of C satisfies $z \ge x$.

$\therefore x = \sup C$.

Thus, finite, nonempty sets always contain their sups.

If C is the empty set, any number is a lower bound for C, so that there is no greatest lower bound, so the inf would not exist.

$\therefore$ No set satisfies all the requirements. $\blacksquare$

Example 1.3.2c Find a bounded subset S of $\mathbb{Q}$ that contains its sup but not its inf.

Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

$\sup S = 1$, $\inf S = 0$.
$\quad\;\in S \qquad\quad \notin S$.

Other useful Properties of $\mathbb{R}$.

① Density of $\mathbb{Q}$ in $\mathbb{R}$.

$\forall\, a, b \in \mathbb{R}$ s.t. $a < b$, $\exists\, r \in \mathbb{Q}$ s.t. $a < r < b$.

Note: Also, between any two rational numbers, there is an irrational real #.

② Nested Interval Property.

Let $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$

Be an infinite set of nonempty intervals, where

$$a_1 \le a_2 \le a_3 \le \cdots$$
$$b_1 \ge b_2 \ge b_3 \cdots$$

Then

$$\bigcap_{n=1}^{\infty} [a_n, b_n] \text{ is } \underline{\text{nonempty}}.$$

$$\parallel$$

$$\{x \in \mathbb{R} : x \in [a_n, b_n]\ \forall n \in \mathbb{N}\}.$$

In fact, $\bigcap_{n=1}^{\infty} [a_n, b_n] = [\sup\{a_n : n \in \mathbb{N}\}, \inf\{b_n : n \in \mathbb{N}\}]$

(Proof in book)

$a_n \le b_k \quad \forall\, n, k.$

because $a_n \le a_{\max\{n,k\}}$
$\le b_{\max\{n,k\}} \le b_k$

$\therefore$ $\inf b_k$ exists and is an upper bound for $\{a_n\}$.

$\therefore$ $\sup\{a_n : n \in \mathbb{N}\} \le \inf\{b_k : k \in \mathbb{N}\}$.

Question: Is the nested interval property true for infinitely large intervals?

$$[a_1, \infty) \supseteq [a_2, \infty) \supseteq [a_3, \infty) \supseteq \dots$$

Does this imply $\bigcap_{n=1}^{\infty} [a_n, \infty)$ is nonempty?

Answer: Sometimes the result is nonempty.

eg if $\sup\{a_n : n \in \mathbb{N}\}$ exists, then

$$\bigcap_{n=1}^{\infty} [a_n, \infty) = [\sup a_n, \infty).$$

But if $\{a_n\}$ has no upper bound, then the intersection is empty.

eg $[1,\infty)\supseteq[2,\infty)\supseteq[3,\infty)\ldots$

and $\bigcap_{n=1}^{\infty}[n,\infty)=\emptyset$.

<u>Example</u> (1.4.4)

Let $a,b\in\mathbb{R}$ with $a<b$.

Let $T=\mathbb{Q}\cap[a,b]$. Prove $\sup T=b$.

<u>Pf.</u> Observe that $\forall x\in T$,

$x\in[a,b]$, so $x\le b$.

Thus b is an upper bound for T.

Can we use the sup lemma?

$\forall\varepsilon$, find $t\in T$ s.t. $b-\varepsilon<t$

yes by Density of $\mathbb{Q}$ in $\mathbb{R}$.

$\forall\varepsilon>0$, $\exists\, r\in\mathbb{Q}$ s.t.

$b-\varepsilon<\max\{a,b-\varepsilon\}<r<b$,

(since $a<b$ and $b-\varepsilon<b$).

Then $r\in\mathbb{Q}$ and $a<r<b$, so $r\in T$.

By the sup lemma, $b = \sup T$. ∎